Class 11 Maths Chapter 9. Sequences and Series

1. Sequence: Sequence is a function whose domain is a subset of natural numbers. It represents the images of 1, 2, 3,… ,n, as f_1, f_2, f_3, …., f_n , where $f_n = f(n)$.

2. Real Sequence: A sequence whose range is a subset of R is called a real sequence.

3. Series: If a_1, a_2, a_3 , … , a_n is a sequence, then the expression $a_1 + a_2 + a_3 + \dots + a_n$ is a series.

4. Progression: A sequence whose terms follow certain rule is called a progression.

5. Finite Series: A series having finite number of terms is called finite series.

6. Infinite Series: A series having infinite number of terms is called infinite series.

Arithmetic Progression (AP)

A sequence in which the difference of two consecutive terms is constant, is called Arithmetic Progression (AP).

Properties of Arithmetic Progression

(i) If a sequence is an AP, then its nth term is a linear expression in n, i.e., its nth term is given by An + B, where A and B are constants and A = common difference.

(ii) nth Term of an AP If a is the first term, d is the common difference and / is the last term of an AP, then

(a) nth term is given by $1= a_n = a + (n – 1)d$

(b) nth term of an AP from the last term is $a'_n = l – (n – 1)d$

(c) $a_n + a'_n = a + 1$
i.e., nth term from the start + nth term from the end
= constant
= first term + last term

(d) Common difference of an AP
$d = T_n – T_{n-1}, \forall\ n > 1$

(e) $T_n = 1/2[T_{n-k} + T_{n+k}], k < n$

(iii) If a constant is added or subtracted from each term of an AP, then the resulting sequence is an AP with same common difference.

(iv) If each term of an AP is multiplied or divided by a non-zero constant k, then the resulting sequence is also an AP, with common difference kd or d/k where d = common difference.

(v) If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an AP, then $2a_{n+1} = a_n + a_{n+2}$.

(vi) (a) Any three terms of an AP can be taken as a – d, a, a + d.

(b) Any four terms of an AP can be taken as a-3d,a- d, a + d, a + 3d.

(c) Any five terms of an AP can be taken as a-2d,a – d, a, a + d, a + 2d.

(vii) **Sum of n Terms of an AP**

(a) Sum of n terms of AP, is given by $S_n = n/2[2a + (n – 1)d] = n/2[a + l]$

(b) A sequence is an AP, iff the sum of n terms is of the form $An^2 + Bn$, where A and B are constants. Common difference in such case will be 2A.

(c) $T_n = S_n – S_{n-1}$

(viii) a^2, b^2 and c^2 are in AP.

$$\Leftrightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in AP.}$$

(ix) If $a_1, a_2, \dots, a_n$ are the non-zero terms of an AP, then

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \dots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$$

(x) Arithmetic Mean

(a) If a, A and b are in AP, then A= (a + b)/2 is called the 2 arithmetic mean of a and b.

(b) If a_1, a_2, a_3 , an are n numbers, then their AM is given by,

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = \frac{na+b}{n+1}$$

$$A_2 = a + 2d = \frac{(n-1)a+2b}{n+1}$$

$$\dots\dots\ A_n = a + nd = \frac{a+nb}{n+1}$$

(c) If a, A_1 , A_2 , A_3 ,…,A_n, b are in AP, then A_1, A_2, A_3,…, A_n are n arithmetic mean between a and b, where

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = \frac{na+b}{n+1}$$

$$A_2 = a + 2d = \frac{(n-1)a+2b}{n+1}$$

$$\ldots\ldots A_n = a + nd = \frac{a+nb}{n+1}$$

(d) Sum of n AM's between a and b is nA
i.e., $A_1 + A_2 + A_3 + + = nA$

Important Results on AP

(i) If $a_p = q$ and $a_q = p$, then $a_{p+q} = 0, T_r = p + q - r$

(ii) If $pT_p = qT_p$, then $a_{p+q} = 0$

(iii) If $a_p = \frac{1}{q}$ and $a_q = \frac{1}{p}$, then $a_{pq} = 1$

(iv) If $S_p = q$ and $S_q = p$, then $S_{p+q} = -(p+q)$

(v) If $S_p = S_q$, then $S_{p+q} = 0$

Geometric Progression (GP)

A sequence in which the ratio of two consecutive terms is constant is called GP. The constant ratio is called common ratio (r).
i.e., $a_{n+1}/a_n = r, \forall\ n \geq 1$

Properties of Geometric Progression (GP)

(i) nth Term of a GP If a is the first term and r is the common ratio

(a) nth term of a GP from the beginning is $a_n = ar^{n-1}$
(b) nth term of a GP from the end is $a'_n = l/r^{n-1}$, l = last term
(c) If a is the first term and r is the common ratio of a GP, then the GP can be written as a, ar, ar^2,… , ar^{n-1}, …
(d) The nth term from the end of a finite GP consisting of m terms is ar^{m-n}, where a is the first term and r is the common ratio of the GP.
(e) $a_n a'_n = al$ i.e., nth term from the beginning x nth term from the end = constant = first term x last term.

(ii) If all the terms of GP be multiplied or divided by same non-zero constant, then the resulting sequence is a GP with the same common ratio.

(iii) The reciprocal terms of a given GP form a GP.

(iv) If each term of a GP be raised to same power, the resulting sequence also forms a GP.

(v) If the terms of a GP are chosen at regular intervals, then the resulting sequence is also a GP.

(vi) If $a_1, a_2, a_3, \dots, a_n$ are non-zero, non-negative term of a GP, then

(a) GM = $(a_1 a_2 a_3 \dots a_n)^{1/n}$
(b) $\log a_1, \log a_2, \log a_3, \dots, \log a_n$ are in an AP and vice-versa.

(vii) If a, b and c are three consecutive terms of a GP, then $b^2 = ac$

(viii) (a) Three terms of a GP can be taken as a/r, a and ar.
(b) Four terms of a GP can be taken as a/r^3, a/r, ar and ar^3.
(c) Five terms of a GP can be taken as a/r^2, a/r, ar and ar^2.

(ix) Sum of n Terms of a GP

(a) Sum of n terms of a GP is given by
(a) Sum of n terms of a GP is given by

$$S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r}, & \text{if } |r| < 1 \\[2mm] \dfrac{a(r^n-1)}{r-1}, & \text{if } |r| > 1 \\[2mm] an, & \text{if } |r| = 1 \end{cases}$$

(b) $S_n = \dfrac{a - lr}{1-r}$ or $S_n = \dfrac{lr - a}{r-1}, r \neq 1$

where l = last term of the GP

(c) If $|r| < 1$, then

$$S_\infty = \frac{a}{1-r}$$

If $|r| \geq 1$, then it does not exist.

(x) Geometric Mean (GM)

(a) If a, G, b are in GP, then G is called the geometric mean of a and b and is given by G = √ab
(b) If a, G_1, G_2, G_3, , G_n, b are in GP, then $G_1, G_2, G_3, \dots, G_n$, are in GM's between a and b,

where

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

....

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

(c) Product of n GM's, G_1 X G_2 X G_3 X … X G_n = G^n

Important Results on GP

(i) If $a_p = x$ and $a_q = y$, then

$$a_n = \left(\frac{x^{n-q}}{y^{n-p}}\right)^{\frac{1}{p-q}}$$

(ii) If $a_{m+n} = p$ and $a_{m-n} = q$, then

$$a_m = \sqrt{pq} \text{ and } a_n = p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$$

(iii) If a, b and c are the pth, qth and rth terms of a GP, then

$$a^{q-r} \times b^{r-p} \times c^{p-q} = 1$$

(iv) nth term of $b + bb + bbb + \ldots$ is

$$a_n = \frac{b}{9}(10^n - 1); b = 1, 2, \ldots, 9$$

(v) nth term of $0{\cdot}b + 0{\cdot}bb + 0{\cdot}bbb + \ldots$ is

$$a_n = \frac{b}{9}(1 - 10^{-n}); b = 1, 2, \ldots, 9$$

Harmonic Progression (HP)

A sequence a_1, a_2, a_3 ,…, a_n of non-zero numbers is called a Harmonic Progression (HP), if the sequence $1/a_1$, $1/a_2$, $1/a_3$, …, $1/a_n$ is an AP.

Properties of Harmonic Progression (HP)

(i) nth term of HP, if $a_1, a_2, a_3, \ldots, a_n$ are in HP, then
(a) nth term of the HP from the beginning

$$a_n = \frac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \frac{a_1 a_2}{a_1 + (n-1)(a_1 - a_2)}$$

(b) nth term of the HP from the end

$$a_n' = \frac{1}{\frac{1}{a_n} - (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)}$$

(c) $$\frac{1}{a_n} + \frac{1}{a_n'} = \frac{1}{a} + \frac{1}{a_n} = \text{constant}$$

$$= \frac{1}{\text{first term of AP}} + \frac{1}{\text{last term of AP}}$$

(d) $a_n = 1/a+(n-1)d$ are the first term and common difference of the corresponding AP.
(ii) Sum of harmonic progression does not exist.

Harmonic Mean

(i) If a, H,b are in HP, then H is called the harmonic mean of a and b i.e., H= 2ab/(a + b)

(ii) If a, $H_1, H_2, H_3, \ldots, H_n$, b are in HP, then
$H_1, H_2, H_3, \ldots, H_n$

are n harmonic means between a and b where

$$H_1 = \frac{(n+1)ab}{a+nb},$$
$$H_2 = \frac{(n+1)ab}{2a+(n-1)b}, \ldots\ldots$$

(iii) Harmonic Mean (HM) between $a_1, a_2, a_3, \ldots, a_n$ is given by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)$$

Important Results on HP

(i) If in a HP, $a_m = n$ and $a_n = m$,

then $a_{m+n} = \frac{mn}{m+n}, a_{mn} = 1, a_p = \frac{mn}{p}$

(ii) If in a HP, $a_p = qr$ and $a_q = pr$

then $a_r = pq$

(iii) If H is HM between a and b, then

(a) $(H-2a)(H-2b) = H^2$

(b) $\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}$

(c) $\frac{H+a}{H-a} + \frac{H+b}{H-b} = 2$

Properties of AM, GM and HM between Two Numbers

If A, G and H are arithmetic, geometric and harmonic means of two positive numbers a and b, then

(i) $A=(a+b)/2$, $G=\sqrt{ab}$, $H=(2ab)/(a+b)$

(ii) $A \geq G \geq H$

(iii) A, G, H are in GP and $G^2 = AH$

(iv) If A,G,H are AM, GM and HM between three given numbers a, b and c, then the equation on having a, b and c as its root is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0,$$

where $A = \frac{a+b+c}{3}, G = (abc)^{1/3}$

and $\frac{1}{H} = \left(\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3}\right)$

(v) If A1,A2 be two AM's, G_1, G_2 be two GM's and H_1, H_2 be two HM's between two numbers a and b, then

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

(vi) If A,G and H be AM, GM and HM between two numbers a and b, then

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\begin{cases} A, & \text{if } n=0 \\ G, & \text{if } n=-\frac{1}{2} \\ H, & \text{if } n=-1 \end{cases}$$

Arithmetico-Geometric Progression

A sequence in which every term is a product of a term of AP and GP is known as arithmetico-geometric progression.

The series may be written as

$$a,(a+d)r,(a+2d)r^2,(a+3d)r^3,\ldots,[a+(n-1)d]r^{n-1}$$

Then,

$$S_n=\frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{\{a+(n-1)d\}r^n}{1-r}, \text{ if } r\neq 1$$

$$S_n=\frac{n}{2}[2a+(n-1)d], \text{ if } r=1$$

$$S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}, \text{ if } |r|<1$$

Sum of Arithmetico-Geometric Series

Type 1 Let $a_1 + a_2 + a_3 + \ldots$ be a given series. If $a_2 - a_1, a_3 - a_2, \ldots$ are in AP or GP, then a_n and S_n can be found by the method of difference.

Let $S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$

$S_n = \quad a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$

So, $S_n - S_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + (a_n - a_{n-1}) - a_n$

$\Rightarrow \quad a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$

$\therefore \quad a_n = a_1 + T_1 + T_2 + T_3 + \ldots + T_{n-1}$

where $T_1, T_2, T_3, \ldots$ are terms of new series and $S_n = \Sigma a_n$

Type 2 It is not always necessary that the series of first order of differences i.e., $a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}$ is always either in AP or in GP in such case.

Let $a_1 = T_1, a_2 - a_1 = T_2, a_3 - a_2 = T_3, \ldots, a_n - a_{n-1} = T_n$

So, $\quad a_n = T_1 + T_2 + \ldots + T_n \quad \ldots(i)$

$\quad a_n = \quad T_1 + T_2 + \ldots + T_{n-1} + T_n \quad \ldots(ii)$

On subtracting Eq. (i) from Eq. (ii), we get

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$ is series of second order of differences and when it is either in AP or in GP, then $a_n = a_1 + \Sigma T_r$

Otherwise, in the similar way, we find series of higher order of differences and the nth term of the series.

Exponential Series

The sum of the series is denoted by the $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty$ number e.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

(i) e lies between 2 and 3.
(ii) e is an irrational number.

(iii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$

(iv) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \infty$

Exponential Theorem

Let a>0, then for all real value of x,

$$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \ldots \infty$$

Logarithmic Series

(i) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$

$\therefore \quad \log_e(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

(ii) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \infty$

$\Rightarrow -\log_e(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \infty$

(iii) $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty\right)$

(iv) $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \infty$

Important Points to be Remembered

(i) In the exponential series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$, all terms carry positive signs whereas in the logarithmic series $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$, the terms are alternatively positive and negative sign.

(ii) In the exponential series, the denominators of the terms involve factorials of natural numbers. But in the logarithmic series the terms do not contain factorials.

(iii) The exponential series is valid for all the values of x. The log series is valid when $|x| < 1$.

Important Result and Useful Series

1. $\sum_{n=0}^{\infty} \frac{1}{n!} = e = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{(n-k)!} = e$

2. $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e - 1$

3. $\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty = e - 2$

4. $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e - 1$

5. $\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty = e - 2$

6. $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots = \frac{e + e^{-1}}{2} = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!}$

7. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots = \frac{e - e^{-1}}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

8. $e^{ax} = 1 + \frac{(ax)}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots + \frac{(ax)^n}{n!} + \ldots \infty$

9. $\sum_{n=0}^{\infty} \frac{n}{n!} = e = \sum_{n=0}^{\infty} \frac{n}{n!}$

10. $\sum_{n=0}^{\infty} \frac{n^2}{n!} = 2e = \sum_{n=1}^{\infty} \frac{n^2}{n!}$

11. $\sum_{n=0}^{\infty} \frac{n^3}{n!} = 5e = \sum_{n=1}^{\infty} \frac{n^3}{n!}$

12. $\sum_{n=0}^{\infty} \frac{n^4}{n!} = 15e = \sum_{n=1}^{\infty} \frac{n^4}{n!}$

13. $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

14. $\sum_{r=1}^{n} k a_r = k \sum_{r=1}^{n} a_r$

15. $\sum_{r=1}^{n} k = k + k + \ldots n \text{ times} = k_n$

16. $\sum_{r=1}^{n} r = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

17. $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

18. $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

19. $\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$

ote and many more)

20. $2 \sum_{i<j=1}^{n} a_i a_j = (a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)$

24. If number of terms in AP/GP/HP are odd, then AM/GM/HM of first and last term in middle term of progression.

25. If pth, qth and rth term of geometric progression are also in geometric progression.

26. If a,b and c are in AP and also in GP, then a=b=c

27. If a, b and c are in AP, then xa, xb and xc are in geometric progression.